estimated by knowing the vibrational energy difference of the colliding pair.

References

[1] White, D. R. and Millikan, R. C., "Vibrational Relaxation in Air," *AIAA Journal*, Vol. 2, No. 10, Oct. 1964, pp. 1844–1846.

[2] Taylor, R. L., Camac, M., and Feinberg, R. M., "Measurements of Vibration-Vibration Coupling in Gas Mixtures," *11th Symposium on Combustion*, The Combustion Institute, 1967, pp. 49–65.

[3] Sato, Y., Tsuchiya, S., and Kuratani, K., "Shock Wave Study of Vibrational Energy Exchange Between Diatomic Molecules," *The Journal of Chemical Physics*, Vol. 50, 1969, pp. 1911–1919.

[4] Millikan, R. C. and White, D. R., "Vibrational Energy Exchange Between N_2 and CO. The Vibrational Relaxation of Nitrogen," *The Journal of Chemical Physics*, Vol. 39, 1963, pp. 98–101.

[5] Schwartz, R. N., Slawsky, Z. I., and Herzfeld, K. F., "Calculation of Vibrational Relaxation Times in Gases," *The Journal of Chemical Physics*, Vol. 20, 1952, pp. 1591–1599.

[6] White, D. R., "Shock-Tube Study of Vibrational Exchange in N_2-O_2 Mixtures," *The Journal of Chemical Physics*, Vol. 49, 1968, pp. 5472–5476.

[7] Breshears, W. D. and Bird, P. F., "Effect of Oxygen Atoms on the Vibrational Relaxation of Nitrogen," *The Journal of Chemical Physics*, Vol. 48, 1968, pp. 4768–4773.

[8] White, D. R. and Millikan, R. C., "Vibrational Relaxation of Oxygen," *The Journal of Chemical Physics*, Vol. 39, 1963, pp. 1803–1806.

Laminar Boundary Layers with Zero Wall Shear, Large Suction, and Strong Adverse Pressure Gradients

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The solutions to the equations for similar laminar boundary layers with zero wall shear have been developed by several authors.[1,2] Most recently, Fox[3] has performed numerical calculations for these so-called "separated profiles" when blowing or suction is present at the wall. In constructing the solution for large suction, Fox confirmed the early result of Prandtl[4] that the pressure gradient parameter must be proportional to the square of the suction rate in order that the wall shear be zero. In an attempt to find the constant of proportionality and to develop higher-order corrections, Fox simply patched together (at a somewhat arbitrary location) the asymptotic expansion for large values of η with the convergent series expansion valid near the wall. The result derived using this procedure appears to contain an error in functional form as will be seen below.

In the present Note a more formal asymptotic expansion procedure is used to develop solutions with large suction when the wall shear is absent. Besides providing a theoretical analogue to Fox's numerical results, the present calculation is interesting because it deals with solutions for large negative values of the pressure gradient parameter. Little is known about the properties of such solutions beyond that given by the strictly numerical results of Libby and Liu,[5] Steinauer,[6] and Rogers[7] in the absence of wall mass transfer.

The describing equations and boundary conditions are[3]

$$f''' + ff'' + \beta(g - f'^2) = 0 \tag{1a}$$

$$g'' + fg' = 0 \tag{1b}$$

subject to

$$f = f_w, f' = 0, f'' = 0, g = g_w \text{ at } \eta = 0 \tag{1c}$$

$$f' = 1, g = 1, \eta \to \infty \tag{1d}$$

where the independent variable is the usual similarity variable η.

There are six boundary conditions for this fifth-order system which, however, contains the parameter β, considered here to be unknown. Hence one aspect of the ensuing calculation will be to find the functional relationship, $\beta = \beta(\mathcal{E})$, where the asymptotic parameter will be $f_w = \mathcal{E}^{-1} \gg 1$.

It is convenient to apply the elementary transformation $f = \mathcal{E}^{-1} + F$ in order to place the parameter $\mathcal{E}$ in the equations rather than in the boundary conditions. The resulting system is

$$F''' + (\mathcal{E}^{-1} + F)F'' + \beta(\mathcal{E})(g - F'^2) = 0 \tag{2a}$$

$$g'' + (\mathcal{E}^{-1} + F)g' = 0 \tag{2b}$$

subject to

$$F = F' = F'' = 0, g = g_w, \text{ at } \eta = 0 \tag{2c}$$

$$F' = 1, g = 1, \text{ at } \eta \to \infty \tag{2d}$$

A solution based on the limit process η fixed, $\mathcal{E} \to 0$ will be constructed first. Assuming that the basic functional relationship $\beta \sim f_w^2 \sim \mathcal{E}^{-2}$ given by Prandtl and confirmed by the numerical analysis of Fox is correct (subsequent calculations verify this point), we consider as the basic approximation to Eqs. (2a) and (2b) for $\mathcal{E} \ll 1$

$$g_0 = F'^2_0 \tag{3a}$$

$$g'_0 = 0 \tag{3b}$$

These equations produce solutions which are capable of satisfying boundary conditions only for large η. It also may be observed that there are no higher-order corrections. Hence the complete outer solutions are simply

$$g \sim 1, F' \sim 1$$

The outer limit process leads only to a trivial result.

In order to develop the complete structure of the boundary layer there must be used another limit process whose form is suggested by applying Eq. (2a) at the surface $\eta = 0$ to show that $F'''(0) = -\beta(\mathcal{E})g_w$ and hence that in the vicinity of the wall there must be a balance of viscous and pressure gradient forces. The transformations

$$H = F\mathcal{E}^{-1}, s = \eta\mathcal{E}^{-1}, G = g$$

lead to the desired formulation. Assuming in addition that $\beta = K_0\mathcal{E}^{-2} + K_1\beta_1(\mathcal{E}) + K_2\beta_2(\mathcal{E}) + \ldots$, where

$$\lim_{\mathcal{E}\to 0} \mathcal{E}^2\beta_1(\mathcal{E}) = 0, \lim_{\mathcal{E}\to 0} (\beta_{n+1}/\beta_n) = 0$$

and where the K_n are to be determined, we find the basic equations are

$$H''' + (1 + \mathcal{E}^2H)H'' + [K_0 + K_1\mathcal{E}^2\beta_1(\mathcal{E}) + K_2\mathcal{E}^2\beta_2(\mathcal{E}) + \ldots](G - H'^2) = 0 \tag{5a}$$

$$G'' + (1 + \mathcal{E}^2H)G' = 0 \tag{5b}$$

$$H = H' = H'' = 0, G = g_w \text{ at } s = 0 \tag{5c}$$

$$H' = 1, G = 1, s \to \infty \tag{5d}$$

where Eqs. (5d) are derived from a matching condition

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according to the limit process, s fixed, $\mathcal{E} \to 0$. It is apparent that this inner system contains sufficient information to lead to uniformly valid solutions. The asymptotic expansions which are used to represent the solution are of the general form $Q = \Sigma_0 \mu_n(\mathcal{E}) Q_n(s)$, where $\lim(\mu_{n+1}/\mu_n) = 0$.

The zeroth order equation for G is

$$G''_0 + G'_0 = 0,\ G_0(0) = g_w,\ G_0(\infty) = 1$$

which has the elementary solution

$$G_0 = 1 + (g_w - 1)e^{-s} \tag{6}$$

Equation (6) is identical to the "asymptotic" result derived by Fox.[3] It follows that the basic momentum equation is

$$H'''_0 + H''_0 - K_0 H'^2_0 = -K_0[1 + (g_w - 1)e^{-s}] \tag{7a}$$

$$H_0(0) = H'_0(0) = H''_0(0) = 0 \tag{7b}$$

$$H'_0(\infty) = 1 \tag{7c}$$

Equation (7) contains four boundary conditions for a third-order system. Hence K_0 may be determined. It should be recalled that $K_0 < 0$ because $\beta(\mathcal{E})$ is large and negative. It may be noted that Eq. (7a) is essentially a special case of the general equation describing the class of similar, two-dimensional laminar boundary layers as noted in Ref. 8 (Chap. V, Eq. (137) where $\alpha = 0, \gamma = 1, \beta = K_0$).

A straightforward numerical analysis for the determination of $H_0(s)$ and K_0 quickly exposes a nonuniqueness problem requiring careful examination of the asymptotic behavior for $s \to \infty$. It is found that there are three different forms of behavior of $H_0(s \to \infty)$ depending upon the value at K_0. Thus for $H(s \to \infty) \simeq s + k + h_0(s)$, there is obtained

$$h_0(s) = -\tfrac{1}{2}(g_w - 1)e^{-s} + A_1 e^{-(\frac{1}{2}-\lambda)s} + A_2 e^{-(\frac{1}{2}+\lambda)s},\ -\tfrac{1}{8} < K_0 < 0 \tag{8a}$$

$$h_0(s) = -\tfrac{1}{2}(g_w - 1)e^{-s} + B_1 e^{-s/2} + B_2 s e^{-s/2},\ K_0 = -\tfrac{1}{8} \tag{8b}$$

$$h_0(s) = -\tfrac{1}{2}(g_w - 1)e^{-s} + e^{-s/2}[C_1 \sin\lambda s + C_2 \cos\lambda s],\ K_0 < -\tfrac{1}{8} \tag{8c}$$

where $\lambda = \frac{1}{2}(1 + 8K_0)^{1/2}$. These forms are a special case of the result given in Ref. 8 [Chap. V, Eq. (145)].

Equations (8a–8c) indicate the nature of the nonuniqueness problem; for a selected K_0 any numerical solution which gives for some s, $H'_0 \simeq 1$ can be continued analytically to infinity by selection of k and of the appropriate pair of arbitrary constants, A_1, A_2, or B_1, B_2 or C_1, C_2. Thus, to determine a unique value of K_0, additional considerations must be made about the asymptotic behavior of solutions for $s \to \infty$. First, the oscillatory solutions for $K_0 < -\frac{1}{8}$ are considered inappropriate because the numerical solutions of Fox indicate nonoscillatory behavior for large s. We note that this banishment may be valid only for the first solution branch given by Fox and considered here; oscillatory behavior may be desired for the other branches which would develop with suction starting from the zero mass transfer, separation profiles found by Libby and Liu.[5,9]

In choosing between the remaining cases in Eqs. (8a) and (8b) it will be asserted that the useful solution is that which decays most rapidly to the specified asymptotic boundary condition.[8] More specifically, only the behavior of the homogeneous terms must be considered because of the conditions on K_0. It may be observed that all of the homogeneous terms decay exponentially. Hence, there is no formal justification for discarding any of the constants A_1, A_2, B_1, B_2 (i.e., if one term was algebraic in nature, the matching condition could be used to show that the relevant constant must be zero). It follows that the most useful solution is that in which the combination of homogeneous terms decays at a maximum rate. The slowest decay rate for the case $K_0 = -\frac{1}{8}$ is $0(se^{-s/2})$. For all larger K_0, $-\frac{1}{8} < K_0 < 0$ the solutions in Eq. (8a) always contain a slower rate inherent in the term $\exp[-(\frac{1}{2} - \lambda)s]$. Hence, the over-all decay rate of the latter case is always slower than the former, thereby justifying the use of the former solution. This choice gives excellent agreement with the numerical results of Fox. Finally, it is noted that the choice $K_0 = -\frac{1}{8}$ implies that the only effect of g_w on $\beta(\mathcal{E})$ arises from the next order term, i.e., for $H_1(s)$, again in agreement with the results of Fox.

Table 1 Values of the second coefficient

g_w	$-K_1$
0	3.36
0.5	3.41
1.0	3.47
1.5	3.51
2.0	3.55

We now consider the next term in the series, i.e., $\beta_1(s)$. Since the first-order corrections in Eq. (5) are $0(\mathcal{E}^2)$, it is quite unlikely that the $\mathcal{E}^2\beta_1(\mathcal{E}) = 0(\mathcal{E})$ as assumed by Fox. It is rather more logical to assert that $\beta_1 = 1$ so that the correction in the third term of Eq. (5a) is $0(\mathcal{E}^2)$. The resulting first-order system is

$$G''_1 + G'_1 = -H_0 G'_0,\ G_1(0) = G_1(\infty) = 0 \tag{9a}$$

$$H'''_1 + H''_1 - 2K_0 H'_0 H'_1 = -K_0 G_1 - H_0 H''_0 - K_1(G_0 - H'^2_0) \tag{9b}$$

$$H_1(0) = H'_1(0) = H''_1(0) = 0,\ H'_1(s \to \infty) = 0 \tag{9c}$$

We have integrated these equations with $K_0 = -\frac{1}{8}$ to provide the values at K_1 shown in Table 1.

With the values of K_0 and K_1 determined, we have for $\beta(\mathcal{E})$ the two term approximation

$$-\beta(\mathcal{E}) \simeq (f_w^2/8) - K_1 \tag{10}$$

This may be compared to the formula of Fox, $-\beta = 0.11 f_w^2 + 0.44 f_w$; although this latter appears to be functionally incorrect, the leading term is remarkably close to Eq. (10).

We find Eq. (10) to be in good agreement with Fox's numerical results although for his largest value of f_w, namely 20, he appears to be concerned with numerical accuracy. Consider two examples showing this agreement: for $g_w = 0$, $f_w = 10$, Fox gives $\beta = -15.848$ whereas Eq. (10) yields -15.86. For $g_w = 1$, $f_w = 20$, Fox gives $\beta = -55.638$ whereas Eq. (10) yields $\beta = -53.47$.

It is noted that the basic solution $f(\eta)$ is not the classical asymptotic suction profile.[8] The classical result would appear if the pressure parameter were $0(1)$ with respect to the limit $f_w \to \infty$. However, in the present problem, the specified boundary condition $f''(0) = 0$ implies that $\beta = \beta(f_w) \to -\infty$. Thus, the approximation that ordinarily leads to the asymptotic suction profile, i.e., $f''' + f_w f'' = 0$, is not valid.

References

[1] Morduchow, M., "Review of Theoretical Investigations on Effects of Heat Transfer on Laminar Separation," *AIAA Journal*, Vol. 3, No. 8, Aug. 1965, pp. 1377–1385.

[2] Spalding, D. B., "Mass Transfer Through Laminar Boundary Layers—1. Velocity Boundary Layers," *International Journal of Heat and Mass Transfer*, Vol. 2, 1961, pp. 15–32.

[3] Fox, H., "Separation Solutions for the Similar Laminar Boundary Layer," *AIAA Journal*, to be published.

[4] Schlichting, H., *Boundary Layer Theory*, 6th ed., McGraw-Hill, New York, 1968, Chap. XIII.

[5] Libby, P. A. and Liu, T. M., "Some Similar Laminar Flows Obtained by Quasilinearization," *AIAA Journal*, Vol. 6, No. 8, Aug. 1968, pp. 1541–1548.

[6] Steinauer, J., "Similar Solutions for the Laminar Wall Jet in a Decelerating Outer Flow," *AIAA Journal*, Vol. 6, No. 11, Nov. 1968, pp. 2198–2200.

[7] Rogers, D., "Reverse Flow Solutions for Compressible Laminar Boundary Layer Equations," *The Physics of Fluids*, Vol. 12, 1969, pp. 517–523.

[8] Rosenhead, L., *Laminar Boundary Layers*, Chap. V, pp. 245–252.

[9] Libby, P. A. and Liu, T. M., "Further Solutions of the Falkner-Skan Equation," *AIAA Journal*, Vol. 5, No. 5, May 1967, pp. 1040–1042.

Magneto-Hypersonic Boundary-Layer Interactions

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Nomenclature

B_0 = applied transverse magnetic field
L = characteristic length = $M_1^2\mu\omega/\rho_1 u_1$
M = Mach number
m = $2(1 - \text{Log } 2)/(\gamma - 1)$; γ = specific heats ratio[3]
P^* = pressure ratio = P_2/P_1
u = x component of velocity; x being the surface coordinate
σ = electrical conductivity
ρ = density
ξ = x/L
η = δ/L
δ = boundary-layer thickness
μ = viscosity coefficient
REX = $\rho_1 u_1 x/\mu_1$

Subscripts

1 = $x = +\infty$ on the boundary-layer surface
2 = $x = x$ on the boundary-layer surface
ω = surface of the plate

IN Ref. 1 to study the magneto-hypersonic viscous interactions for the insulated flat plate, the hydromagnetic compressible boundary-layer equations of Meyer[2] for the case of no induced magnetic field were investigated by a generalized von Kármán Integral method. Corresponding to the nonmagnetic momentum integral equation of Pai[3] the magnetic equation can be derived and the additional term involved is obtained as $-(\sigma B_0^2 M_2^2 L/2\rho_2 u_2)(P^*\eta)$. Since the fluid characteristics involving the subscript 2, in an interaction phenomena are unknown, an approximation made in Ref. 1 was that $(M_2^2|\rho_2 u_2) \approx (M_1^2/\rho_1 u_1)$ which then converts the unknown magnetic parameter into the square of the Hartmann number, in terms of the characteristics with subscript 1, which are known.[3] But this approximation, which implies that $P^* = 1$ made the analysis of Ref. 1 valid for the weak interaction case only. In fact, it can be seen from the present analysis that no such approximation is needed for discussing the magneto viscous interactions of the hypersonic boundary layer flow. Following Ref. 3 the magnetic momentum integral equation under the hypersonic conditions can be derived as,

$$(2/m) = d/d\xi(\eta^2 P^*) - (\sigma B_0^2 L/m)(M_2^2/\rho_2 u_2)(\eta^2 P^*) \quad (1)$$

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Since $u_2/u_1 \backsim 1$ for the hypersonic case,

$$(M_2^2/\rho_2 u_2) = (u_2/\gamma P_2) = u_1(u_2/u_1)/\gamma P_1(P_2/P_1) \approx (u_1/\gamma P_1)(1/P^*) = (M_1^2/\rho_1 u_1)(1/P^*)$$

So that Eq. (1) becomes

$$(2/m) = d/d\xi(\eta^2 P^*) - (\sigma B_0^2 L/m)(M_1^2/\rho_1 u_1)(\eta^2) \quad (2)$$

But

$$(\sigma B_0^2 L . M_1^2/\rho_1 u_1) = (\sigma B_0^2 L^2/\mu\omega) = H_m^2$$

Where H_m is the Hartmann number and hence Eq. (2) simplifies to

$$(2/m) = d/d\xi(\eta^2 P^*) - (H_m^2/m)\eta^2 \quad (3)$$

Equation (3), being independent of the approximation of Ref. 1, is valid for both strong and weak interaction cases and can be solved for the corresponding cases by using the corresponding pressure distributions.

For the strong case, P^* can be taken as,

$$P^* = \{\gamma(\gamma + 1)M_1^2/2\}(d\eta/d\xi)^2 \quad (4)$$

Substituting relation (4) in Eq. (3) and observing as in Ref. 3 that near the origin,

$$\eta \cong (32/9mc)^{1/4}\xi^{3/4} \quad (5)$$

Where $c = \gamma(\gamma + 1)M_1^2/2$, a series solution for η may be written as

$$\eta = \zeta^3 \sum_0^\infty b_m \zeta^m \quad (6)$$

where $\zeta = \xi^{1/4}$ and the rest is as in Ref. 3.

For the weak case, if it is assumed that $P^* = 1$ can be put in (3) then a simple solution can be obtained as

$$\eta^2 = \{e^{(H_m^2/m)\xi} - 1\}(2/H_m^2) \quad (7)$$

which for small ξ implies that $\eta \propto \xi^{1/2}$, which may as well mean that there is no viscous interaction at all. Similarly if $P^* = 1$ used in (7) be replaced by $P^* = 1 + \gamma M_1 d\eta|d\xi$ then the analysis of Eq. (3) will show that $\eta \propto \xi^{2/3}$. Hence,

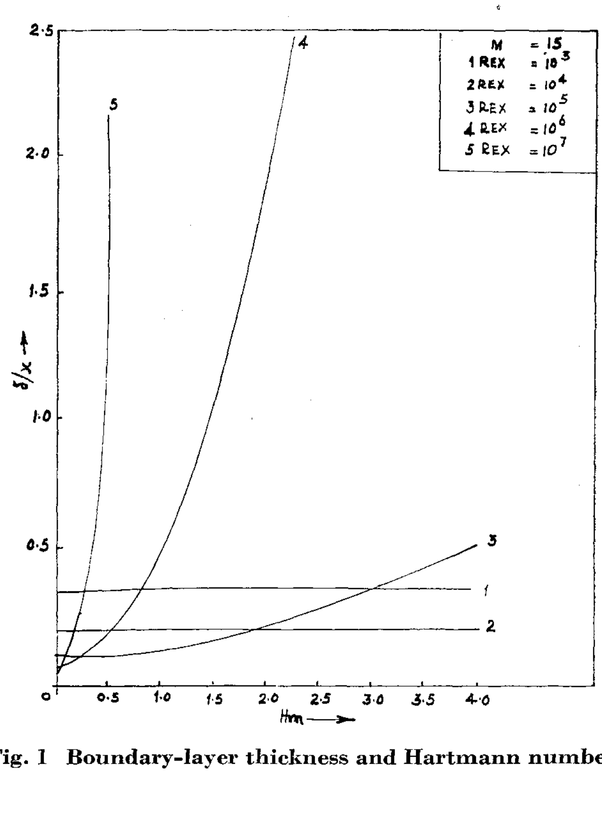

Fig. 1 Boundary-layer thickness and Hartmann number.